Published by Institute of Physics Publishing for SISSA

Received: *December 22, 2006*
Accepted: *February 21, 2007*
Published: *March 6, 2007*

Computing amplitudes in topological M-theory

Giulio Bonelli and Alessandro Tanzini

International School of Advanced Studies (SISSA) and INFN Sezione di Trieste, via Beirut 2-4, 34014 Trieste, Italy
E-mail: bonelli@sissa.it, tanzini@sissa.it

Maxim Zabzine

Department of Theoretical Physics, Uppsala University, Box 803, SE-751 08 Uppsala, Sweden
E-mail: Maxim.Zabzine@teorfys.uu.se

Abstract: We define a topological quantum membrane theory on a seven dimensional manifold of G_2 holonomy. We describe in detail the path integral evaluation for membrane geometries given by circle bundles over Riemann surfaces. We show that when the target space is $CY_3 \times S^1$ quantum amplitudes of non-local observables of membranes wrapping the circle reduce to the A-model amplitudes. In particular for genus zero we show that our model computes the Gopakumar-Vafa invariants. Moreover, for membranes wrapping calibrated homology spheres in the CY_3, we find that the amplitudes of our model are related to Joyce invariants.

Keywords: Topological Field Theories, Topological Strings, M-Theory.

Contents

1. Introduction

Topological strings on Calabi-Yau (CY) manifolds describe a certain sector of the superstrings. In particular, various BPS quantities in superstrings can be computed using the topological version. There are two different topological string models on CY, A- and B-models, which can be obtained from the physical model through a topological twist [35]. Different superstring theories are unified in the context of M-theory, which is expected to correspond to a supermembrane theory, yet unknown. Thus, in analogy with topological strings, one may expect that there exists the topological version of M-theory which should capture the BPS sector of M-theory. In particular it is natural to expect that the topological version of M-theory should count membrane instantons [10] which induce nonzero corrections to the superpotential. The notion of topological M-theory[1] has been proposed in [22] (also for earlier proposal see [24]). The idea is that the topological M-theory should

[1] Indeed, A.Losev [30] has advocated the idea of M-theory in the context of the topological string even before these works.

provide the unifying description of A- and B-models on $CY_3 \times S^1$. In general the theory should be defined over seven dimensional G_2 manifolds. In [22] the analysis has been done at the classical level of effective actions. Different arguments in favor of topological M-theory have been proposed [25, 7, 32].

Our goal is to propose a microscopic description of topological M-theory in term of topological membrane theory. This work is the continuation of our previous paper [16] (also see the earlier work [15]). Different attempts with a similar aim can be found in [4, 5, 28].

In our attempt we closely follow the analogy with A-model which is a topological sigma model [34] coupled to two dimensional gravity. For the computation in the zero genus sector it is enough to consider the topological sigma model since there are no gravitational moduli for S^2. The topological sigma model can be thought of as the BRST quantization [6] of the following topological action

$$S_{\text{top}} = \int\limits_{\Sigma_2} X^*(K), \tag{1.1}$$

where K is the Kähler form on the CY target. We would like to construct a membrane analog of the topological sigma model. We propose that on G_2 manifolds[2] this model arises as the BRST quantization of the topological action

$$S_{\text{top}} = \int\limits_{\Sigma_3} X^*(\Phi), \tag{1.2}$$

where Φ is the canonical three form on the G_2 manifold. Naively the action (1.2) should reduce to (1.1) for wrapped configurations on $CY_3 \times S^1$ [15] and this was our initial motivation for the choice of (1.2).

Indeed, contrary to the standard two dimensional sigma model, the BRST quantization of the action (1.2) for generic Σ_3 is a difficult task. Actually, setting a fully covariant gauge fixing seems to couple the membrane to gravitational moduli in a non trivial way. This implies two technical problems. The first is that ghost and antighost sectors are unbalanced and it is difficult to us to find natural projectors to fix it in a generic situation. The second concerns coupling 3D topological gravity in an appropriate way. Without doubt this is a hard problem and presently we do not have a complete understanding of it.

However we are able to carry out the program in the case where Σ_3 is an S^1- bundle over a Riemann surface Σ_g (Seifert manifold[3]). Indeed the resulting theory counts associative maps, which correspond to membrane instantons on G_2 manifolds.

Hopefully the present work puts on firmer ground some results which were derived in on shell models by Harvey and Moore [26] and by Beasley and Witten [8]. Actually we give a clear and complete definition of quantum amplitudes via an off-shell procedure. Among other results, we confirm the prescription of calculating the Euler characters of the moduli space of associative cycles given by Beasley and Witten [8].

[2] In this paper we consider the topological membrane theory defined on a seven manifold of G_2 holonomy. However, the G_2 holonomy condition can be relaxed to a closed G_2-structure, see further details in [16].

[3] Actually the notion of Seifert manifold is more general and includes also orbifold generalizations [9]. Here for simplicity we focus on Seifert manifolds admitting a free U(1) action.

JHEP03(2007)023

A related issue has to do with the interpretation of our theory as the microscopic description of topological M-theory. The topological membrane theory that we propose in this paper has the nice feature of reducing to the A model on a Calabi-Yau threefold CY_3 once considered on seven manifolds[4] of type $S^1 \times CY_3$. This will be established via equivariant localization of the membrane path-integral for Seifert geometries. We show that the *non local* membrane equivariant observables reduce to *local* A model observables. Moreover, our reduction to the A model directly reproduces Gopakumar-Vafa invariants.

Actually, it is only the winding sector of the membrane which gets reduced to the A model. The unwinding sector instead stays as a topological theory of membranes in CY_3 which should represent the nonperturbative completion of the A model itself. Once evaluated on CY_3 whose cycles are rational homology spheres, this sector generates Joyce torsion invariants [29]. These are obtained from a very simple path-integral argument.

Another proposal for a microscopic description of topological M theory has been suggested by de Boer et al. [20, 21] in terms of topological strings with G_2 target space. The closed string model on $CY_3 \times S^1$ reduces in this case to the B model. This suggests to interpret the conjectured S-duality in topological strings as an electro-magnetic duality in topological M theory, which exchanges strings and membranes in seven dimensions.

The plan of the paper is the following. In section 2 we will describe covariantly membrane instantons and discuss a few related classical issues. In section 3 we discuss the membrane theories for Seifert three dimensional geometries with generic G_2 holonomy target. In section 4 we define and calculate the membrane path integral reducing it to an integral over the moduli space of associative maps. In section 5 we comment about the observables and their quantum amplitudes. In section 6 we study wrapped sector of the model on $S^1 \times CY_3$. We discuss how our calculation is related to the zero genus Gopakumar-Vafa invariants. In section 7 we calculate the membrane on CY_3 and show the relation with Joyce invariants and torsion factors. Section 8 is left for few concluding comments.

2. Covariant membrane instantons on G_2-manifolds

In this section we give a covariant description of membrane instantons in seven dimensional manifolds with G_2-structure. These are the calibrated three-cycles. Membranes are embeddings of reference geometries in the target, that is $X : \Sigma \to M$. Given an invariant closed 3-form Φ on a Riemannian seven manifold M, then the cycle $X(\Sigma)$ is calibrated [27] if the volume element with respect to the induced metric and the pullback of Φ on the cycle coincide.

In [16] a non covariant description of membrane instantons was given as solutions of the equation

$$\dot{X}^\mu \pm \Phi^\mu{}_{\nu\rho}\partial_1 X^\nu \partial_2 X^\rho = 0, \tag{2.1}$$

where the membrane world-volume is of the form $\Sigma_2 \times S^1$ (also S^1 can be replaced by either an interval or a real line) with Σ_2 being a Riemann surface. In [16] it has been proved that

[4] This generalizes easily to seven manifolds $S^1 \times M_6$, where M_6 has SU(3) structure.

JHEP03(2007)023

the above equation is equivalent to the calibration condition

$$d\ \mathrm{vol}(\Sigma) = \mp\frac{1}{6}X^*(\Phi). \tag{2.2}$$

In this section we will give a natural covariant description of membrane instantons which is valid for any topology. Let us consider the solutions to the equation

$$dX^\mu \pm \Phi^\mu{}_{\nu\rho} *_\Sigma dX^\nu \wedge dX^\rho = 0, \tag{2.3}$$

where $*_\Sigma$ is the Hodge dual w.r.t. a given world-volume metric on the membrane $h_{\alpha\beta}$. It is straightforward to show that the above equations imply the calibration condition (2.2) and the equation for the induced world-volume metric is

$$h_{\alpha\beta} = \partial_\alpha X^\mu g_{\mu\nu}\partial_\beta X^\nu \equiv \gamma_{\alpha\beta} \quad . \tag{2.4}$$

In order to show this, let us multiply (2.3) by the world-volume Hodge star, getting in components[5]

$$\sqrt{h}\ \epsilon_{\alpha\beta}{}^{\gamma}\partial_\gamma X^\mu \pm \Phi^\mu{}_{\nu\rho}\partial_\alpha X^\nu\partial_\beta X^\rho = 0, \tag{2.5}$$

where $h_{\alpha\beta}$ is a generic world-volume metric. Then by multiplying (2.5) by $g_{\sigma\mu}\partial_\delta X^\sigma$ one gets

$$\gamma_{\alpha\beta} = \mp\frac{h_{\alpha\beta}}{\sqrt{h}}\ X^*(\Phi), \tag{2.6}$$

which is the relation between the auxiliary and induced metrics. For a generic non-constant map X and non-degenerate auxiliary world-volume metric $h_{\alpha\beta}$ this relation is non-singular.

Moreover, by multiplying (2.5) by $\Phi_{\mu\sigma\tau}\partial_\delta X^\sigma\partial_\eta X^\tau$ and using the vector cross product identity[6]

$$(\partial_\alpha X \times \partial_\beta X)\cdot(\partial_\delta X \times \partial_\eta X) = \gamma_{\alpha\delta}\gamma_{\beta\eta} - \gamma_{\alpha\eta}\gamma_{\beta\delta} \tag{2.8}$$

one gets

$$\sqrt{h}\ \epsilon_{\alpha\beta}{}^{\gamma}\epsilon_{\gamma\delta\eta}X^*(\Phi) \pm (\gamma_{\alpha\delta}\gamma_{\beta\eta} - \gamma_{\alpha\eta}\gamma_{\beta\delta}) = 0, \tag{2.9}$$

which upon using

$$\epsilon_{\alpha\beta}{}^{\gamma}\epsilon_{\gamma\delta\eta} = \frac{1}{h}\left(h_{\alpha\delta}h_{\beta\eta} - h_{\alpha\eta}h_{\beta\delta}\right) \tag{2.10}$$

and (2.6) gives the calibration condition (2.2). Then plugging back the calibration condition in (2.6) one finally gets the induced metric equation (2.4). To check that the system is not overdetermined, then one just checks that the sum of all the squares of (2.5) does not give further conditions.

Let us now comment on the relationship between the covariant (2.3) and non covariant (2.1) equations. The covariant equations read in components

$$\partial_0 X^\mu \pm \Phi^\mu{}_{\nu\rho}\frac{\sqrt{h}}{2}\ \epsilon_0{}^{ab}\partial_a X^\nu\partial_b X^\rho = 0\ , \tag{2.11}$$

$$\partial_a X^\mu \pm \Phi^\mu{}_{\nu\rho}\sqrt{h}\ \epsilon_a{}^{b0}\partial_b X^\nu\partial_0 X^\rho = 0,\ . \tag{2.12}$$

[5]We work in the conventions where $\epsilon_{012} = 1$.

[6]Notice that

$$\Phi^{\mu\nu\rho}\Phi_{\mu\sigma\tau} = \delta^{[\nu\rho]}_{\sigma\tau} - *\Phi^{\nu\rho}_{\sigma\tau} \tag{2.7}$$

.

where the $a, b = 1, 2$ indexes run over the spatial components. For the rest of the paper we choose sign plus in the associative map equation.

Assuming that the world-volume of the membrane is of the form $\Sigma_3 = S^1 \times \Sigma_2$, one can partially fix the diffeomorphism symmetry of (2.3) by setting

$$\begin{aligned} h_{00} &= \det \hat{h}_{ab} \ , \\ h_{0a} &= 0 \end{aligned} \tag{2.13}$$

Then by inserting (2.11) into (2.12) we get the induced metric condition $\gamma_{\alpha\beta} = \hat{h}_{\alpha\beta}$. Finally, using this condition it is easy to verify that the time component of the covariant equation (2.11) reduces precisely to (2.1).

This shows that the non-covariant equations (2.1) follow from (2.3) in the case $\Sigma_3 = S^1 \times \Sigma_2$. We underline that for membranes with this topology it is possible to fix diffeomorphisms in such a way that the induced metric *decouples* and the membrane instanton equations can be written without reference to any world-volume metric. This is a relevant issue for the quantization of the membrane as we will further comment in the next section.

Finally, let us notice that a completely analogous construction can also be designed for calibrated four-cycles for an eight dimensional manifold with Spin(7)-structure.

2.1 Local structure

Let us study the local structure of the solutions of our instanton equations. As we have just proved, the image of associative maps spans associative cycles. The purpose of this subsection is to show this from a complementary constructive point of view.

The local G_2 manifold geometry in the vicinity of an associative 3-cycle M_3 is given by the total space of the spin bundle over M_3 [18, 31]. Adapting the coordinate choice to the local geometry as $\{x^\mu\} = \{x^\alpha, x^A\}$ with $\alpha = 1, 2, 3$ and $A = 1, 2, 3, 4$, the 3-form defining the G_2 structure expands as

$$\Phi = -\sqrt{g_3}\, \epsilon_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma + (\gamma_\alpha)_{AB}\, dx^\alpha \wedge dx^A \wedge dx^B + O(|x^A|),$$

where γ_α are the gamma-matrices in three dimensions satisfying the Clifford algebra $\{\gamma_\alpha, \gamma_\beta\} = 2(g_3)_{\alpha\beta}$. The instanton equations then reduce to

$$*dX^\alpha - \epsilon^\alpha{}_{\beta\gamma}\sqrt{g_3}\, dX^\beta \wedge dX^\gamma = 0, \tag{2.14}$$

$$dX^A + (\gamma_\alpha)^A{}_B\, dX^\alpha \wedge dX^B = 0. \tag{2.15}$$

The first equation (2.14) is identically satisfied since X^α span a covering of the associative cycle $\Sigma \to M_3$. The second (2.15) is solved by maps $X^A = 0$ up to linear deformations described by

$$*\nabla \delta X^A + (\gamma_\alpha)^A{}_B dX^\alpha \wedge \nabla \delta X^B = 0.$$

Solutions of the above equation correspond to zero modes of the twisted McLean Dirac operator [31].

In these adapted coordinates, our model makes contact with some previous discussions on the semiclassical quantization of membranes [26, 8].

JHEP03(2007)023

3. Gauge-fixing the membrane

3.1 Trying to gauge fix the general model

In this section we consider the gauge fixing for the following topological membrane theory

$$S_{\text{top}} = -\frac{1}{6}\int_{\Sigma_3} X^*(\Phi), \tag{3.1}$$

where Φ is the closed three form associated to a G_2-structure on a seven dimensional manifold M_7. We start by discussing the covariant instanton description of the previous section as a gauge fixing. The topological gauge symmetry of the action is

$$\delta X^\mu = \epsilon^\mu. \tag{3.2}$$

The corresponding BRST operator s is defined as follows

$$sX^\mu = \Psi^\mu, \qquad s\Psi^\mu = 0, \tag{3.3}$$

where $\Psi^\mu \in X^*(TM) \otimes \Omega^0(\Sigma)$ is the ghost associated to ϵ^μ and has ghost number one. In order to fix this symmetry, we could choose the following covariant gauge function

$$\mathcal{F}^\mu_\alpha = \partial_\alpha X^\mu + \frac{\sqrt{h}}{2}\epsilon_\alpha^{\beta\gamma}\Phi^\mu{}_{\nu\rho}\partial_\alpha X^\nu \partial_\beta X^\rho = 0. \tag{3.4}$$

We observe that the square of (3.4) gives rise to the bosonic action

$$\begin{aligned} S_{\text{bos}} &= -\frac{1}{6}\int X^*(\Phi) + \frac{1}{3}\int d^3\sigma\sqrt{h}h^{\alpha\beta}\frac{1}{2}\mathcal{F}^\mu_\alpha g_{\mu\nu}\mathcal{F}^\nu_\beta \\ &= \frac{1}{6}\int d^3\sigma\sqrt{h}\left[h^{\alpha\beta}\gamma_{\alpha\beta} + \frac{1}{2}\left(h^{\alpha\beta}\gamma_{\alpha\beta}\right)^2 - \frac{1}{2}h^{\alpha\beta}\gamma_{\beta\gamma}h^{\gamma\delta}\gamma_{\delta\alpha}\right]. \end{aligned} \tag{3.5}$$

The stationary value of S_{bos} with respect to variations of $h_{\alpha\beta}$ is reached for $h_{\alpha\beta} = \gamma_{\alpha\beta}$ at

$$S_{\text{bos}}[h=\gamma] = \int d^3\sigma\sqrt{\gamma} \tag{3.6}$$

that is the Nambu-Goto action. Henceforth, (3.5) could be regarded as a generalization for the membrane of the Polyakov action.

In order to implement the covariant gauge-fixing (3.4) one has to introduce a couple of antighosts and lagrangian multipliers $(\bar{\Psi}_{\alpha\ \mu}, B_{\alpha\ \mu})$ with ghost numbers $(-1,0)$ respectively. These fields are sections of the bundle $X^*(T^*M)\otimes\Omega^1(\Sigma)$. The ghost and antighost sector of our theory are therefore unbalanced. This follows from the fact that the covariant gauge fixing (3.4) we choose is actually redundant and extra gauge conditions have to be implemented in the antighost sector, either via the introduction of ghosts-for-ghosts or of suitable projection operators.

This is similar to what happens in the topological sigma model case, where indeed the antighosts are constrained by using the complex structure projectors [34, 6, 13]. However, in the topological membrane case a further complication takes place, since in the covariant equations for associative cycles (2.3) the calibration conditions (2.2) cannot be fully disentangled from the induced metric conditions (2.4). This suggests that for a fully covariant description of the membrane one could need the coupling with topological gravity.

Nonetheless, for membranes whose world-volume is a circle fibration over a Riemann surface (Seifert manifolds) one can easily construct suitable projection operators on the antighost sector and write a completely gauge fixed sigma model. Since this covers a wide class of physically relevant three-dimensional manifolds, as for example rational homology spheres, we find it interesting to focus on this case.

3.2 Gauge-fixed action for Seifert membranes

A Seifert manifold $S_{g,p}$ is a three-dimensional manifold given by the total space of a circle bundle over a Riemann surface Σ_g

$$\begin{array}{ccc} S_{g,p} & \longleftarrow & S^1 \\ {\scriptstyle \pi}\big\downarrow & & \\ \Sigma_g & & \end{array} \tag{3.7}$$

the $\mathrm{U}(1)$ group acts as rotations on the fibers of (3.7). The topology of $S_{g,p}$ is completely specified in terms of the genus g of the Riemann surface and the degree p of the bundle. The Seifert manifolds admit a globally defined non-vanishing invariant one-form κ, which can be taken to be a connection of the principal $\mathrm{U}(1)$ bundle with curvature

$$\mathrm{vol}_\omega d\kappa = p\ \pi^*\omega, \tag{3.8}$$

where ω is a symplectic form on Σ_g, such that $\int_{\Sigma_g}\omega = \mathrm{vol}_\omega$. The dual of κ is the fundamental vector field k which generates the $\mathrm{U}(1)$ action on $S_{g,p}$ and satisfies $i_k\kappa = 1$. From these definitions it follows

$$\int_{S_{g,p}} \kappa\ d\kappa = p. \tag{3.9}$$

The metric on $S_{g,p}$ can be written in the form

$$ds^2 = \pi^* ds^2_{\Sigma_g} + \kappa \otimes \kappa. \tag{3.10}$$

By taking the Hodge dual of κ w.r.t. this metric, one gets the useful relation

$$*\kappa = \pi^*\omega. \tag{3.11}$$

In local coordinates one has $\kappa = dt + a$, where t is a local coordinate on the circle and a is a $\mathrm{U}(1)$ connection on the base. Notice that the above splitting is defined up to local gauge transformations

$$t \to t + \lambda \quad , \qquad a \to a - d\lambda, \tag{3.12}$$

while κ stays invariant.

Thanks to the existence of κ, one can easily perform a projection on the antighost sector in order to remove the redundancy of the covariant gauge-fixing condition (2.3). To this purpose, it is enough to project the one-forms $(\bar{\Psi}_{\mu\alpha}, B_{\mu\alpha})$ along κ, or equivalently, to rewrite the gauge fixing condition as

$$\mathcal{F}^\mu = (*\kappa) \wedge dX^\mu + \Phi^\mu{}_{\nu\rho}\, \kappa \wedge dX^\nu \wedge dX^\rho \quad . \tag{3.13}$$

In (3.13) we moved the Hodge star operator on the first term to simplify the subsequent computations. From now on we do not write explicitly $\wedge$-product and it is assumed to be present in appropriate places. To the gauge-fixing condition (3.13) we associate a couple of antighosts and Lagrangian multipliers $(\bar{\Psi}_\mu, B_\mu)$ which are zero-forms on the world-volume, such that the ghost and antighost sector are now perfectly balanced. The BRST symmetry reads

$$sX^\mu = \Psi^\mu, \qquad s\Psi^\mu = 0, \qquad s\bar{\Psi}_\mu = B_\mu, \qquad sB_\mu = 0, \tag{3.14}$$

and the gauge-fixing action in delta gauge

$$S_{g.f.} = \int\limits_{S_{p,g}} s\, \left[\bar{\Psi}_\mu (*\kappa\; dX^\mu + \Phi^\mu{}_{\nu\rho}\; \kappa\; dX^\nu dX^\rho)\right]. \tag{3.15}$$

By evaluating explicitly this action on manifolds with G_2 holonomy, one gets

$$S_{g.f.} = \int\limits_{S_{p,g}} B_\mu \mathcal{F}^\mu - \bar{\Psi}_\mu \left(*\kappa\nabla\Psi^\mu + 2\Phi^\mu{}_{\nu\rho}\; \kappa\; dX^\nu \nabla\Psi^\rho + \Psi^\sigma \Gamma_\sigma{}^\mu{}_\lambda \mathcal{F}^\lambda \right). \tag{3.16}$$

The last term in (3.16) seems to spoil general covariance on the target space. Actually as it is well explained in [13], the covariant BRST transformations can be obtained by a simple field redefinition, which is suggested already by (3.16) to be $B'_\mu = B_\mu - \Gamma^\lambda{}_{\sigma\mu} \bar{\Psi}_\lambda \Psi^\sigma$.

By integrating by parts and using $d * \kappa = 0$, which follows from (3.11), we can symmetrize the fermionic kinetic operator of (3.16) as

$$S^{\text{kin}}_{\text{ferm}} = -\frac{1}{2} \int\limits_{S_{p,g}} \left(\bar{\Psi}_\mu D \Psi^\mu + \Psi^\mu D^\dagger \bar{\Psi}_\mu \right), \tag{3.17}$$

where

$$D \equiv *\kappa\nabla + 2\Phi^\mu{}_{\nu\rho}\; \kappa\; dX^\nu \nabla, \tag{3.18}$$

$$D^\dagger \equiv *\kappa\nabla + 2\Phi^\mu{}_{\nu\rho}\; \kappa\; dX^\nu \nabla + 2\Phi^\mu{}_{\nu\rho}\; d\kappa\; dX^\nu - 2\Phi^\mu{}_{\nu\rho}\; \kappa\; \nabla dX^\nu. \tag{3.19}$$

Notice that $D = \delta\mathcal{F}/\delta X|_{\mathcal{F}=0}$ so that the D zero-modes span the tangent directions to the moduli space of associative cycles. Moreover in the presence of fermionic zeromodes the δ-gauge is clearly not complete. In the next sections, while evaluating the membrane path integral, we will improve it to the general case.

3.3 Hamiltonian formalism on Seifert geometries and relation with the Nambu-Goto membrane

Because of the presence of the fundamental vector field k, Seifert manifolds are naturally suited for Hamiltonian quantization. In fact it is easy to check that the first term of (3.13) can be rewritten as

$$*\kappa\; dX^\mu = \sqrt{h_S}\;\; \mathcal{L}_k X^\mu\; d^3\sigma, \tag{3.20}$$

where h_S is the determinant of the Seifert metric (3.10). Hence the first term of the projected gauge-fixing conditions (3.13) is the Lie derivative of the embedding map X w.r.t. the fundamental vector field k, which is the natural generalization of the "time" derivative for Seifert manifolds. In particular, for trivial circle bundles it is immediate to recognize that (3.13) reduces to the non-covariant gauge-fixing (2.1). In this subsection we discuss the Hamiltonian formalism for the Nambu-Goto membrane and show how this relates with the topological model discussed so far.

We then start from the Nambu-Goto action (3.6) and compute the momentum associated with $\mathcal{L}_k X$

$$p_\mu = \frac{\delta S_{NG}}{\delta(\mathcal{L}_k X^\mu)} = \sqrt{\gamma}\ \kappa_\alpha \gamma^{\alpha\beta} g_{\mu\nu} \partial_\beta X^\nu. \tag{3.21}$$

This satisfies the following primary constraints

$$p^2 - \gamma \kappa_\alpha \gamma^{\alpha\beta} \kappa_\beta = 0 \quad \text{and} \quad \mathcal{L}_{v_{(i)}} X^\mu p_\mu = 0, \quad i = 1, 2, \tag{3.22}$$

where the two vectors $v_{(i)}$ are linear independent and orthogonal to κ, that is $i_{v_{(i)}}\kappa = 0$.

The phase space action of the system is

$$S_{ham} = \int\limits_{S_{g,p}} d^3\sigma \left[p_\mu \mathcal{L}_k X^\mu - N \left(p^2 - \gamma \kappa_\alpha \gamma^{\alpha\beta} \kappa_\beta \right) - \sum_{i=1,2} N_{(i)} \mathcal{L}_{v_{(i)}} X^\mu p_\mu \right]. \tag{3.23}$$

To get a gauge-fixed action we can fix the Lagrange multipliers to $N_{(i)} = 0$ and $N = \frac{1}{\omega_{12}}$ in terms of the volume density ω_{12} on the base Riemann surface. As in the usual case the action (3.23) is quadratic in the momenta. We can then integrate them out by fixing them to their classical values. This leads to the Lagrangian action

$$S_{\text{Lag}} = \frac{1}{4} \int\limits_{S_{g,p}} \kappa\omega \left[(\mathcal{L}_k X)^2 + \frac{4}{(\omega_{12})^2} \gamma \left(\kappa_\alpha \gamma^{\alpha\beta} \kappa_\beta \right) \right], \tag{3.24}$$

where $\omega = \omega_{12} d\sigma^1 d\sigma^2$.

Let us now show that in this gauge, up to an additive topological term, (3.24) is the square of our instanton equation. Actually, writing

$$f^\mu = \mathcal{L}_k X^\mu + \Phi^\mu{}_{\nu\rho} \partial_\alpha X^\nu \partial_\beta X^\rho \epsilon^{\alpha\beta\gamma} \kappa_\gamma, \tag{3.25}$$

(where $\epsilon^{123} = 1/\omega_{12}$) we get

$$f^2 = (\mathcal{L}_k X)^2 + \frac{4}{(\omega_{12})^2} \gamma \left(\kappa_\alpha \gamma^{\alpha\beta} \kappa_\beta \right) + \frac{4}{\omega_{12}} X^*(\Phi)_{123}$$

and comparing with (3.24), we get

$$S_{\text{Lag}} = \frac{1}{4} \int_S \kappa\omega f^2 - \int_S X^*(\Phi), \tag{3.26}$$

which shows that associative maps satisfying $f^\mu = 0$ are Nambu-Goto instanton membranes.

3.4 Coupling to gravity

The topological model for the membrane we presented in the previous sections plays the analogous rôle for topological M theory as the topological sigma model for topological strings. The formulation of a complete theory involves the sum over the world-volume metrics $h_{\alpha\beta}$. To this end, one could introduce a gravitational BRST multiplet containing the BRST partner of the metric

$$sh_{\alpha\beta} = \chi_{\alpha\beta} \ , \qquad s\chi_{\alpha\beta} = 0 \ , \tag{3.27}$$

and a suitable gauge-fixing condition. As we already noted in section 2, the covariant instanton equation (2.3) implies also the condition on the world-volume metric (2.4) which could be used as a gravitational gauge-fixing. Indeed we saw that (2.3) is a redundant gauge condition with respect to the symmetry (3.2) and this leads to an unbalanced antighost sector. Unfortunately this problem is not completely fixed by the gravitational topological symmetry since the antighost sector count twenty-one components while the BRST ghost are only seven from (3.3) plus six from (3.27).

Once again the situation is simpler for Seifert manifolds. In this case the very form of the metric (3.10) indicates that the coupling with topological gravity would contain the integration over the moduli space given by the total space of the fibration

$$\begin{array}{ccc} \widehat{\mathcal{M}}_g & \longleftarrow & \mathbf{T^g} \\ \big\downarrow & & \\ \mathcal{M}_g & & \end{array}$$

where $\mathcal{M}_g$ is the moduli space of the Riemann surface Σ_g and $\mathbf{T^g}$ the moduli space of $\mathrm{U}(1)$ flat connections on Σ_g. The above diagram hints about a relation with the Gopakumar-Vafa invariants. Presently we do not know how to treat properly the gravitational moduli for Seifert geometries. However we will be able to give a definite prescription for genus zero, see section 6.

4. Explicit evaluation of the membrane path-integral

In this section we use the topological symmetry to evaluate the membrane path integral explicitly.

Let us start by modifying the gauge fixed action (3.15) as follows

$$S_{\rm top} + s \int\limits_{S_{g,p}} \left(\frac{1}{\beta} \bar{\Psi} \mathcal{F} + \bar{\Psi} g^{-1} B \right) , \tag{4.1}$$

where β is a c-number parameter that will be used to drive the exact semiclassical expansion while the second term in the gauge fermion resolves the fermion zero-mode ambiguity. The metric appearing in the last term is the G_2 holonomy one with respect to which the three-form Φ is covariantly constant.

JHEP03(2007)023

Let us now expand our fields as follows. Denote the full set of solutions of $\mathcal{F}^\mu = 0$ by X_0 and expand

$$X = X_0 + \beta x, \quad \Psi = \Psi_0 + \beta^{1/2}\psi, \quad B = B_0 + b, \quad \bar{\Psi} = \bar{\Psi}_0 + \beta^{1/2}\bar{\psi}, \tag{4.2}$$

where the above decompositions are defined starting from the fermionic kinetic operator $\mathcal{D}$ at the generic associative map X_0, namely $\mathcal{D}_0 = \mathcal{D}|_{X=X_0}$. Then, x and b span the space of non zero modes of $\mathcal{D}_0$ and ψ and $\bar{\psi}$ span a copy with grassmanian statistics of the same space. Moreover, Ψ_0 are zero modes of D_0, $\bar{\Psi}_0$ zero modes of $D_0^\dagger$, while B_0 is a general solution of

$$D_0^\dagger B_0 = \bar{\Psi}_0 \left(\frac{\delta D}{\delta X}\right)|_{X_0}\Psi_0 \ . \tag{4.3}$$

As such it can be written as $B_0 = \tilde{B}_0 + \hat{B}_0$ were $\hat{B}_0$ is a particular solution of (4.3) and $\tilde{B}_0$ is a general solution of the associated homogeneous equation $D_0^\dagger \tilde{B}_0 = 0$.

The path integral measure is then

$$\mathcal{D}[X, \Psi, \bar{\Psi}, B] = d[X_0, \Psi_0, \bar{\Psi}_0, B_0]\mathcal{D}[x, \psi, \bar{\psi}, b]\beta^{\frac{1}{2}TrJ}$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on the space spanned by (x, b). Expanding in non-zero modes of $\mathcal{D}_0$, namely $D_0 x_n = \lambda_n b_n$ and $D_0^\dagger b_n = \lambda_n x_n$ (where $\lambda_n \neq 0$), we see that the b_ns and the x_ns are in one to one correspondence and therefore $TrJ = 0$.

Let us now plug the above expansion in the gauge fixed action (4.1) so to get

$$S_{\text{top}} + \int_{S_{g,p}} \left[bD_0x + \bar{\psi}D_0\psi + B_0^2 + b^2 + R\bar{\Psi}_0^2\Psi_0^2\right] + o(\beta^{1/2}), \tag{4.4}$$

where R stands for the curvature tensor. We can use then the BRST symmetry s to localize our path integral, namely to evaluate it at $\beta \to 0$. In order to further simplify the evaluation of the path integral at $\beta = 0$ we can get rid of the b^2 term in the action by the rescaling $(x, b) \to (\mu x, \mu^{-1}b)$ and send $\mu \to \infty$. Henceforth the integral over the fluctuations of the fields gives simply one because of exact boson-fermion cancellation. This result is clearly expected because of the isomorphism between the ghost and antighost sectors which makes our theory an $N_T = 2$ topological model [14].

The integral over B_0 is Gaussian and gives two contributions: one is a constant overall factor $\int d\tilde{B}_0 \mathrm{e}^{-\tilde{B}_0^2}$. The other gives a further quartic term[7] in the fermionic zero modes $\mathrm{e}^{-\hat{B}_0^2}$. The integral over X_0, namely the space of solutions of $\mathcal{F}^\mu = 0$, is then an integral over the moduli space of associative maps. The different instanton sectors are classified topologically by the homology class

$$N = X_*(\Sigma_3) \in H_3(M, \mathbb{Z}). \tag{4.5}$$

[7] Notice that this is an usual term arising in topological field theories as for example in a twisted version of $\mathcal{N} = 4$ SYM in four dimensions.

Sometimes it is convenient to introduce a basis $[e_i]$ of $H_3(M,\mathbb{Z})$, where $i = 1,\ldots,b_3(M)$. Thus we can expand $N = \sum_{i=1}^{b_3} N_i[e_i]$ and the instanton sectors are labeled by $b_3(M)$ integers N_i. Let us denote the moduli space associative maps as $\mathcal{M}_N$ for $N \in H_3(M)$. The integral over Ψ_0 is therefore along $T\mathcal{M}_N$. Due to the three-dimensional index theorem $dim(Ker\, D_0) = dim(Ker\, D_0^\dagger)$, the integration along $\bar{\Psi}_0$ is along a space of the same dimension. Notice that the remaining terms in the reduced action, namely $R\bar{\Psi}_0^2\Psi_0^2 + \hat{B}_0^2$, are quartic in the fermionic zero modes and therefore soak-up the fermionic zero modes. Therefore performing the integration over zero modes Ψ_0 and $\bar{\Psi}_0$ we get the Euler class

$$e(\mathcal{M}_N) = Pf(\mathcal{R}_N)\,, \tag{4.6}$$

where now $\mathcal{R}_N$ is regarded as Lie algebra valued two-form, curvature two-form on $\mathcal{M}_N$. It is important to stress that we took into account the contribution from $\hat{B}_0^2$-term. Due to the Gauss-Bonnet thorem $e(\mathcal{M}_N)$ upon the integration over $\mathcal{M}_N$ (i.e., the remaining integration $d[X_0]$) gives the Euler number $\chi(\mathcal{M}_N)$ of $\mathcal{M}_N$. Introducing the parameters

$$t_i = \frac{1}{6}\int\limits_{e_i} \Phi \tag{4.7}$$

with Φ being G_2 three form and defining $q_i = e^{-t_i}$ we can write the partition function as follows

$$Z = \sum_{N\in H_3(M)} \chi(\mathcal{M}_N)\; q^N \tag{4.8}$$

where q^N denotes $\prod_{i=1}^{b_3} q_i^{N_i}$. The evaluation of $\chi(\mathcal{M}_N)$ can be a hard problem and depends very much on the structure of $\mathcal{M}_N$ which is not known. However in section 6 we will present the evaluation of $\chi(\mathcal{M}_N)$ for specific N on $CY_3 \times S^1$.

Alternatively the result (4.8) can be argued through more formal Mathai-Quillen formalism (see [36] for the review) with $\mathcal{F}$ being the appropriate zero section. The analysis goes along the lines presented in [14].

5. Observables and moduli spaces

We now discuss the observables of the general topological model introduced in section 3. Let us recall the situation as it was already discussed in [16]. The construction of observables closely follows the analogous one for the A-model topological string. However the path integral evaluation of them takes a different path because of the three dimensional index theorem.

For a nontrivial element $[A] \in H^q(M)$ we can define the following co-cycles

$$\mathcal{C}_i^{q-i} \equiv \frac{1}{i!} A_{\mu_1\ldots\mu_q} dX^{\mu_1}\ldots dX^{\mu_i}\Psi^{\mu_{i+1}}\ldots\Psi^{\mu_q} \tag{5.1}$$

for $q \geq i \geq 0$ and zero otherwise. In (5.1) the upper index stands for the ghost number and the lower index for the degree of the differential form on the world-volume Σ. Using the transformations (3.14) we can derive the decent equations for $\mathcal{C}_i^{q-i}$

$$d\mathcal{C}_{i-1}^{q-i+1} = (q-i+1)s\mathcal{C}_i^{q-i}. \tag{5.2}$$

Thus $\mathcal{C}_0^q$ are BRST-invariant *local* observables labeled by the elements of the de Rham complex $H^\bullet(M)$, while from $\mathcal{C}_i^{q-i}$ with $i > 0$ we can construct BRST-invariant *non-local* observables as integrals

$$\int_{c_i} \mathcal{C}_i^{q-i}, \tag{5.3}$$

where c_i is an i-cycle on Σ.

Not all observables have non-vanishing correlators in the theory. This in general depends on the possible anomalies that the theory displays in the ghost sector. Indeed, the gauge-fixed action (3.15) has at the classical level a ghost number conservation law, with Ψ having ghost number 1, $\bar{\Psi}$ having ghost number -1 and X having ghost number 0. The BRST transformation s (3.14) changes the ghost number by 1.

Notice that all the observables, but $\mathcal{C}_i^0$ with $i = 1, 2, 3$, defined in (5.1) have positive ghost number. Thus, in order to have non-vanishing correlators there should be a compensating ghost number anomaly.

The linearized equations for the fermionic fluctuations around the instanton background are

$$D_0 \Psi_0^\mu = 0, \tag{5.4}$$

$$D_0^\dagger \bar{\Psi}_{0\,\mu} = 0. \tag{5.5}$$

The equation (5.4) is the first order variation of the associative map (3.13). As such, Ψ_0 can be interpreted as a section of the tangent bundle to the moduli space $\mathcal{M}$ of associative maps. The operator $D^\dagger$ is the adjoint of D, and thus the ghost number anomaly is given by the index $ind(D)$. Since our theory lives in three dimensions $ind(D)$ vanishes by index theorem. Thus the correlators of any observable in (5.1) with positive ghost number do not get contribution. Hence we are left with the observables $\int_{c_i} \mathcal{C}_i^0$ with $i = 1, 2, 3$. For example, the case $i = 3$ corresponds to our classical action (3.1) and its variations in $H^3(M)$. As such its evaluation corresponds just to the partition function, which computes the Euler characteristic of the moduli space of associative maps [16]. The situation is very similar to supersymmetric quantum mechanics (for a review, see [13]).

The above conclusions can be enforced also by direct calculation of the relevant quantum amplitudes via the method explained in the previous section. Suppose then we want to evaluate the insertion of any of the observables (5.3) or of a product of them. Actually these are functionals of X and Ψ only $\mathcal{O}(X, \Psi)$. The change of variable (4.2) $X = X_0 + \beta x$ and $\Psi = \Psi_0 + \beta^{1/2}\psi$ and the subsequent $\beta \to 0$ limit leaves $\mathcal{O}(X_0, \Psi_0)$ and the insertion is independent on the fluctuations of the fields. To calculate the quantum amplitude, therefore one stays with

$$\int d[X_0, \Psi_0, \bar{\Psi}_0] \mathrm{e}^{-S_{\mathrm{top}} - \int_\Sigma (R\bar{\Psi}_0^2\Psi_0^2 + \hat{B}_0^2)} \mathcal{O}(X_0, \Psi_0). \tag{5.6}$$

JHEP03(2007)023

Because of the equality among the number of Ψ_0s and $\bar{\Psi}_0$s there is a simple selection rule which states that the only observables which have not a priori zero amplitude are the ones at zero ghost number where there is no dependence upon Ψ at all and therefore no extra Ψ_0 mode to be soaked up.

Let us note that the above conclusions are based on the smoothness of the moduli space of associative maps, that is on the stability of its tangent bundle. This condition might be violated by some components thus invalidating the above arguments. Unfortunately not much is known about the geometry of these moduli spaces (for the recent discussion, see [1, 2]) and therefore we are not able to further discuss this issue.

6. Membranes on S^1 and topological A model

In this section we show the explicit reduction at quantum level of our topological membrane theory to the topological A model on appropriate geometries. Let us consider the target geometry to be $S^1 \times X$, where X is a Calabi-Yau threefold. Therefore, we have $\Phi = dX^7 K + \mathrm{Re}\Omega$, where K is the Kahler two-form and Ω the holomorphic three-form on X. Then, as already discussed in [16], membranes can have zero or non zero winding along the target S^1. The full quantum theory will receive two distinct contributions by these two sectors. The zero-winding sector corresponds to membranes all internal to the CY threefold and will be the subject of next section. The non zero winding sector is our concern now and will be studied in the following. In particular, we show how to recover *local* observables in the A model from *non local* observables in an equivariant sector of the topological membrane theory.

The first simplification in calculating the membrane path integral in the non-zero winding sector comes from the presence of the isometric S^1 direction. Once combined with rotations along the fiber of the Seifert geometry, this allows us to compute the membrane path integral via equivariant localization. Let us first define the equivariant BRST action

$$s_\kappa X^\mu = \Psi^\mu, \quad s_\kappa \Psi^\mu = uk^\alpha \partial_\alpha X^\mu - wV^\mu, \quad s_\kappa \bar{\Psi}_\mu = B_\mu, \quad s_\kappa B_\mu = uk^\alpha \partial_\alpha \bar{\Psi}_\mu \tag{6.1}$$

and $s_\kappa u = s_\kappa w = 0$. The vector $V^\mu = \delta^\mu_7$ generates the target space isometry and $k^\alpha \partial_\alpha$ is the vector generating the Seifert U(1)-action. Notice that s_κ^2 closes on the U(1) action that we will use to localize, namely

$$s_\kappa^2 X^\mu = uk^\alpha \partial_\alpha X^\mu - wV^\mu, \quad s_\kappa^2 \Psi^\mu = uk^\alpha \partial_\alpha \Psi^\mu, \quad s_\kappa^2 \bar{\Psi}_\mu = uk^\alpha \partial_\alpha \bar{\Psi}_\mu, \quad s_\kappa^2 B_\mu = uk^\alpha \partial_\alpha B_\mu. \tag{6.2}$$

We write the above as $s_\kappa^2 = \mathcal{L}_U$ on the field space.

Before entering the path integral evaluation, let us face the problem of defining equivariant observables. As a first point, notice that the topological action (4.1) itself is trivially s_κ-closed and U invariant. Actually, it is not s_κ-exact. As it is clear, equivariant observables are obtained by calculating the cohomology of s_κ in the space of U-invariant functionals. Let us analyze a particular observable

$$\mathcal{O} = \oint_{S^1} A_{\mu\nu\rho} dX^\mu \Psi^\nu \Psi^\rho + f_\mu dX^\mu. \tag{6.3}$$

with $A \in H^3(M_7)$ and $f \in \Lambda^1(M_7)$. By direct calculation we verify that

$$s_\kappa \mathcal{O} = 0 \quad \text{iff} \quad df = 2wi_V A = 2wB, \tag{6.4}$$

where it is enough for f to exist locally. Since $M_7 = S^1 \times X$ we have $i_V A = B \in H^2(X)$ and the solution to the above condition is provided by picking f to be the local potential of B divided by $2w$. Because of the simple connectivity of the CY this local choice can be extended to cover all the image of the fiber S^1 in X and therefore defines unambiguously our observable (6.4). Indeed for any element B of $H^2(X)$ we can define the equivariant observable (6.3) specified by $A = dX^7 \wedge B$. Notice that an analogous construction for the local observable $i_V A\Psi\Psi$ does not hold.

The evaluation of the membrane path integral by equivariant localization under the U(1) action (6.2) reduces the domain of integration to the U-fixed points. These field configurations are the solutions of $\mathcal{L}_U[fields] = 0$, that is

$$X^\mu = \frac{w}{u} V^\mu (t - \sigma) + x^\mu, \quad \Psi^\mu = \psi^\mu, \quad \bar{\Psi}_\mu = \bar{\psi}_\mu, \quad B_\mu = b_\mu, \tag{6.5}$$

where t is a local coordinate on the S^1 fiber of the Seifert and $\sigma = -i \ln e$, e being a section of the U(1) bundle over Σ_g. Notice that when the U(1) bundle is non trivial σ becomes multivalued. Henceforth, in order the map (6.5) to be regular, we require[8] the Seifert to be $S^1 \times \Sigma_g$ and thus we can choose $\sigma = 0$. In (6.5) x^μ, ψ^μ, $\bar{\psi}_\mu$ and b_μ depend only on the base Σ_g. Moreover, using the gauge symmetry of the Seifert geometry (3.12), the field x^7 can be reabsorbed by an appropriate choice of the local coordinate t. Correspondingly, due to the BRST invariance, also the fermionic partner ψ^7 has to be set to zero.

While reducing to the fixed locus, the induced measure is obtained by the product of the weights of the U(1) action (6.2) on the tangent space.[9] This is easily calculated to be

$$\left[\det{}^{1/2}(u\mathcal{L}_\kappa)\right]^{[1+1-1-1]} = 1, \tag{6.6}$$

where the exponents (+1 for the fermions and −1 for the bosons) sum up to zero just for usual fermion/boson cancellation.

Notice that the way we are counting the field modes in the measure has changed w.r.t. sections 4 and 5. In fact there the measure was defined by diagonalizing the three-dimensional fermionic kinetic operator in (3.17). Instead in the equivariant approach we diagonalize the operator $\mathcal{L}_U \otimes \nabla$, where ∇ is the fermionic kinetic operator for the A-model on the base. Since in general the two operators do not commute, these correspond to different regularization schemes of the path integral. In particular the selection rules discussed in sections 4 and 5 have to be changed since now one has to take into account the index theorem for the reduced fermionic kinetic operator ∇.

[8] We underline that this condition is necessary for a target space CY$\times S^1$ but could be relaxed for more general geometries as for example S^1 fibrations over a CY.

[9] Actually, for non isolated fixed points, one should [11] consider also the curvature of the normal bundle $\mathcal{R}_\mathcal{N}$. However, due to the mutual independence of the Fourier modes along the fiber, in our case $\mathcal{R}_\mathcal{N} = 0$.

Let us now proceed further by analyzing the gauge fixed action. As a preliminary step, let us notice that $s_\kappa \int_\Sigma \bar{\Psi}_\mu \mathcal{F}^\mu = s \int_\Sigma \bar{\Psi}_\mu \mathcal{F}^\mu$ and therefore the gauge fixed action stays unchanged. The gauge fixing conditions (3.13) under localization on (6.5), reduce to

$$\mathcal{F}^7|_{f.l.} = \kappa f^7 = \kappa \left[\frac{w}{u}\omega + x^*(K) \right], \tag{6.7}$$

$$\mathcal{F}^m|_{f.l.} = \kappa f^m = \kappa \left[\mathrm{Re}\Omega^m_{np} dx^n dx^p \right] \tag{6.8}$$

and the reduced gauge fixed action reads

$$s_\kappa \int\limits_{\Sigma_g} \bar{\Psi}_\mu \mathcal{F}^\mu|_{f.l.} = \int\limits_{\Sigma_g} b_7 f^7 + b_m f^m + 2\bar{\psi}_7 \; K_{mn} dx^m \nabla \psi^n + \bar{\psi}_m \mathrm{Re}\Omega^m_{np} dx^n \nabla \psi^p, \tag{6.9}$$

where we integrated along the fiber with $\oint dt = 1$.

Let us first integrate over the 7-sector. The constraint (6.7) gives rise to a delta-function which fixes the volume form on the base in terms of the degree of the reduced map and of the non-vanishing S^1 winding. Actually the A-model on Σ_g depends only on conformal classes and therefore has to be independent on the actual choice of ω, which can be considered as a gravitational modulus of the Seifert geometry. Indeed this situation is similar to the discussion of torus partition function in A-model [12]. Henceforth, in order to recover the topological sigma A model we have to integrate on ω and its BRST partner. This integration presents a subtlety related to degenerate volume forms lying on the boundary $\omega = 0$. We assume the existence of a suitable BRST invariant regularization of the ω quantum measure avoiding this point.

The resulting gauge-fixed action reads finally

$$S_{g.f.} = \int\limits_{\Sigma_g} b_m \left(\mathrm{Re}\Omega^m_{np} dx^n dx^p \right) + \bar{\psi}_m \left(\mathrm{Re}\Omega^m_{np} dx^n \nabla \psi^p \right) \quad . \tag{6.10}$$

Let us now show how this reduces to the usual A model path integral. First of all the bosonic constraint in (6.10) can be written in complex coordinates as

$$\Omega^{\bar{I}}_{JK} \partial_z x^J \partial_{\bar{z}} x^K = 0 \tag{6.11}$$

and its complex conjugate. Clearly, since Ω is non-degenerate, (6.11) is equivalent to the (anti-)holomorphicity of the map x. The resulting path integral is

$$\begin{aligned} &\int \mathcal{D}[x,\psi,\bar{\psi}] \delta \left(\mathrm{Re}\Omega dx dx \right) \mathrm{e}^{-\int_{\Sigma_g} \bar{\psi} \mathrm{Re}\Omega dx \nabla \psi} \\ &\quad = \int \mathcal{D}[x,\psi,\bar{\psi}] \mathrm{det}^{-1} \left(\mathrm{Re}\Omega dx \right) \delta \left(dx - J \; * dx \right) \mathrm{e}^{-\int_{\Sigma_g} \bar{\psi} \mathrm{Re}\Omega dx \nabla \psi} + [\mathrm{anti-holom.}], \end{aligned} \tag{6.12}$$

here $dx - J \; * dx = 0$ is the usual holomorphicity condition and the second factor takes into account the equal contribution of anti-holomorphic maps. To make a closer contact with the usual A model, we can redefine the antighost $\bar{\psi}$ as

$$\rho^{\bar{I}}_z = \frac{1}{2} \bar{\psi}^J \Omega^{\bar{I}}_{JK} \partial_z x^K \qquad \rho^I_{\bar{z}} = \frac{1}{2} \bar{\psi}^{\bar{J}} \bar{\Omega}^I_{\bar{J}\bar{K}} \partial_{\bar{z}} x^{\bar{K}} \; . \tag{6.13}$$

This change of variables is singular on constant maps. Actually, as we can see from (6.7) these maps correspond to $\omega = 0$ and thus are avoided in our regularized path integral. The Jacobian associated to (6.13) is compensated by the determinant factor in (6.12) leaving us with

$$\int \mathcal{D}[x, \psi, \rho]\delta\left(dx - J\ \ast dx\right) \mathrm{e}^{-\int_{\Sigma_g} \rho \nabla \psi} + [\text{anti} - \text{holom.}] \tag{6.14}$$

which is the A model path integral in δ-gauge. Indeed on holomorphic maps ρ satisfies exactly the self-duality condition

$$\rho + J\ \ast \rho = 0 \tag{6.15}$$

which is the usual projection on the antighosts of the A model [34].

Let us now consider the evaluation of the equivariant observables (6.3). On the fixed locus (6.5) this reduces to

$$\mathcal{O}|_{f.l.} = \oint_{S^1} \frac{w}{u} dt\ B_{mn}\psi^m \psi^n = \frac{w}{u} B_{mn}\psi^m\psi^n, \tag{6.16}$$

where B is defined below the equation (6.4). So, the *non local observable of the membrane theory* (6.3) *reduces to the local observable of the topological A model* of the standard type. The topological action (3.1) reduces consistently to

$$S_{\text{top}}|_{f.l.} = -\frac{1}{6} \int_{S^1 \times \Sigma_g} K_{mn} \frac{w}{u} dt dx^m dx^n = -\frac{1}{6}\frac{w}{u} \int_{\Sigma_g} x^*(K). \tag{6.17}$$

To summarize, the equivariant localization of our membrane theory on $CY_3 \times S^1$ reproduces the topological A sigma-model. In particular, due to the absence of gravitational moduli, if $\Sigma_g = \mathbf{P}^1$ we can reduce to A-model calculation on CY_3. However, compared with the standard A-model, the contribution of the constant maps is missing here. Indeed in such a case the constraint (6.7) is singular. This should come as no surprise since the map which has non-zero winding along S^1 and is constant on the base does not satisfy the associative map condition.

6.1 Relation to Gromov-Witten and Gopakumar-Vafa invariants

In this subsection we would like to summarize our equivariant calculation for the zero genus and relate it to the general expression (4.8). We comment also on the relation to the Gromov-Witten and Gopakumar-Vafa invariants [23].

Let us first of all remind some basic facts about the zero-genus A-model calculations which will be useful in the following discussion. Consider the case when X is an ideal Calabi-Yau threefold. Following the standard arguments the three point function at zero genus is given by

$$\langle O_i O_j O_k \rangle = \int_X l_i \wedge l_j \wedge l_k + \sum_{C \subset X} \sum_{m=1}^{\infty} n_i n_j n_k\ e^{-m \int_C K}, \tag{6.18}$$

where l_i is the basis in $H^2(X)$ which label the observables O_i and $n_i = \int_C l_i$. In (6.18) we have to sum up over all rational curves C and all possible degrees m. Alternatively the second term in (6.18) can be rewritten as

$$\sum_{0\neq\beta\in H_2(X,\mathbb{Z})} \sum_{m=1}^{\infty} n^0_\beta \; n_i n_j n_k \; q^{m\beta}, \tag{6.19}$$

where n^0_β enumerates rational curves in X of class β. As usual $q^{m\beta}$ is defined as follows

$$q^{m\beta} = e^{-m\beta\cdot\vec{t}}, \qquad \int_\beta K = \beta\cdot\vec{t}. \tag{6.20}$$

The free energy $F_0(q)$ is related through third derivative to $\langle O_i O_j O_k\rangle$

$$\langle O_i O_j O_k\rangle = \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} F_0(q)$$

and thus we obtain

$$F_0(q) = \sum_{0\neq\beta\in H_2(X,\mathbb{Z})} \sum_{m=1}^{\infty} n^0_\beta \frac{1}{m^3} q^{m\beta}, \tag{6.21}$$

where we ignore the classical piece in (6.18). n^0_β are integer numbers which are called zero genus Gopakumar-Vafa invariants. Alternatively we can rewrite (6.21) as

$$F_0(q) = \sum_{0\neq\beta\in H_2(X,\mathbb{Z})} N^0_\beta q^\beta, \tag{6.22}$$

where N^0_β are zero genus Gromov-Witten invariants.

Let us now turn to our equivariant calculation from the previous section 6. We calculate the three point function of non-local observables (6.3) which are labeled by the basis in $H^2(X)$. Now m is the winding number of the membrane,[10] $m = \frac{w}{u}$ in (6.5) of the previous section. The equivariant calculation produces the following contribution to the three point function

$$\sum_{0\neq\beta\in H_2(X)} \sum_{m=1}^{\infty} \; m^3 \; \left(n^0_\beta \; n_i n_j n_k\right) \; q^{m\beta}, \tag{6.23}$$

where the factor m^3 comes from the reduction of the observables (6.16). Integrating these three point correlators to a free energy of wrapped membranes produces the following result

$$\sum_{0\neq\beta\in H_2(X)} \sum_{m=1}^{\infty} \; n^0_\beta \; q^{m\beta}, \tag{6.24}$$

which is just counting the wrapped associative maps and since X is ideal Calabi-Yau, all these maps are isolated. It is natural to compare (6.24) with the general formula (4.8)

[10] We fix the base orientation in (6.7) in such way that m takes only positive values. Switching to negative values corresponds to the consideration of anti-instantons, i.e. the minus sign in our associative map condition (2.3).

from where we see that the Euler number in the case of isolated maps is just the number of maps.

We hope that above reasoning can be generalized to the case when X is not an ideal CY. We believe that the formula (6.24) should be still valid in this case.

7. Membranes on CY's, homology spheres and Joyce invariants

We still have to threat the zero winding sector of the membrane. This corresponds to expand the path integral around $X^7 = 0$ (or any constant point on S^1). Plugging this choice in the associative maps equation, one stays with the analog equation for membranes spanning special lagrangian submanifolds in the CY-threefolds

$$*\kappa g_{ab} X^*(dx^b) + \kappa X^*(i_{\partial_a} \mathrm{Re}\Omega) = 0 \tag{7.1}$$

and the condition $X^*(\omega) = 0$. Actually, the latter is implied by the first since it is just the condition of the cycle being lagrangian. We can use a suitable shift in the gauge fermion to lift completely the X^7, ψ^7 sector.

As it was explained already in [16], one can formulate also a membrane theory on CY threefolds. This theory localizes on the special Lagrangian submanifolds of CY_3, see [16] for further details.

The corresponding instanton equation for Seifert manifolds is given by (7.1). The partition function can be obtained following the same steps we outlined in the previous sections for G_2 manifolds. In particular, one can calculate it in the case in which the associative cycle is a rational homology sphere. This corresponds to a Seifert geometry built on a Riemann sphere. Then we have, for $p \geq 0$, $S_{0,p+1} = S^3/\mathbf{Z}_p$ which generalizes the Hopf bundle $S_{0,1} = S^3$.

Let us consider the case in which $\Sigma = S^3$ and the CY to be such that every SLag is a rational homology sphere $S_{0,p_i+1} = S^3/\mathbf{Z}_{p_i}$ or just restrict our attention to such a geometrical sector as in [29].

In this case, the tangent bundle at each cycle is trivial and therefore the cycles are isolated. Calculating the partition function one gets simply the Euler number of each point-like component of the moduli space, which is 1. Actually, in summing up to isomorphisms one has to be careful not to get an under-counting of each component. In fact, two maps which differ by a $\mathbf{Z}_{p_i}$ transformation map to the same cycle, but are counted as distinct. Actually, in the orbifolded angular direction θ, the maps

$$\theta \to n\theta + 2\pi \frac{q}{p_i}, \quad q = 0, \dots, p_i - 1$$

identify the same target cycle. Therefore for any such a cycle we get a multiplicity of $|\mathbf{Z}_{p_i}| = p_i = |H^1(S_{0,p_i+1}, \mathbf{Z})|$.

Henceforth the partition function on the CY manifold X is given by

$$Z^X_{\mathrm{membrane}} = \prod_{i=1,\dots,h_3(X)} |H^1(S_{0,p_i+1}, \mathbf{Z})| \sum_{n_i \in \mathbf{N}} e^{-t_i n_i} c(n_i), \tag{7.2}$$

where $\{t_i\}$ span the real part of the complex moduli of X, n_i is the degree of the map at the i-th cycle and $c(n_i)$ is a function of such a degree that we do not calculate. In fact, these coefficients would arise by calculating the integrals over the moduli space of multiple covering associative maps. Unfortunately very little is known about such spaces.

The torsion prefactor $|H^1(S_{0,p_i+1}, \mathbf{Z})|$ is in agreement with a conjecture by D. Joyce [29]. In this paper, he also conjectures a prescription to account for multiple coverings with $c(n_i) = \frac{1}{n_i}$.

Actually, the same reasoning can be applied to membranes in G_2 manifolds with topology $S^3/\mathbf{Z}_p \times \mathbf{R}^4$ and produces the same torsion prefactor in agreement with [26], although in such a case the multiple covering coefficient is expected to be $c(n) = 1/n^2$ [33, 17].

8. Summary and open problems

We presented a full quantum description of topological membranes with Seifert geometry mapping into a seven manifold of G_2 holonomy. The theory is localized on associative maps and the partition function calculates the Euler number of the moduli space of these maps. Thus the contribution of the constant maps gives zero partition function since the Euler number of a seven manifold is always zero. Considering the manifold $CY_3 \times S^1$ the topological membrane has two sectors, wrapping membranes and unwrapping membranes. Using equivariant localization we have shown that the wrapped sector reduces to the A-model calculation, except the constant map contribution. In particular for genus zero we show that our model computes the Gopakumar-Vafa invariants. The unwrapped sector corresponds to the theory localized on special Lagrangian submanifolds inside CY_3. In section 7 we studied in detail this sector for the case of rational homology spheres getting the Joyce invariants. Actually this result can be interpreted as a counting of distinct D-branes in A-model, namely flat $\mathrm{U}(1)$ bundles over SLags. This confirms that the topological membrane describes both the perturbative and non-perturbative sector of topological strings. Concerning its second quantized description, the absence of a non-trivial contribution by the constant maps prevents us from recovering an Hitchin functional. Notice in fact that our microscopic description applies to a strongly coupled regime with respect to the topological G_2 string proposal by de Boer et al. [20], whose restriction to the constant maps gives the Hitchin functional. Thus indeed in our calculations this functional has not to arise at all. However we believe that the proper interpretation as an effective action for topological M-theory [22] should arise in a second quantized membrane theory.

Several open problems remain to be analyzed for a deeper understanding of the membrane dynamics. It would be useful to develop a viable fully covariant gauge fixed model. As we discussed in sections 2 and 3 this is a relevant issue for a wider comprehension of the coupling to 3D gravity. In this direction also the relation of our topological model with usual super-membrane formulations via a suitable twist should be investigated. Actually we have shown that when written in adapted local coordinates around the associative cycles, our model reproduces the super-membrane theories developed by Harvey and Moore [26] and Beasley and Witten [8] in order to compute the membrane contributions to the su-

JHEP03(2007)023

perpotential in G_2 M-theory compactifications. To proceed further in these computations within our framework, one should be able to take into account multi-covering maps.

Another interesting aspect to investigate concerns the geometry of the moduli space of associative cycles and its relation with the membrane quantum amplitudes.

A natural generalization of the present approach to the topological three-brane on Spin(7) geometries is to choose the four manifold on which the three-brane is modeled to be an S^1 fibration over a three manifold. The program we outlined in this paper can be repeated in this case too. Moreover, the index theorem does not imply the vanishing of the ghost number anomaly and the situation is much similar to the case of the topological A model in two dimensions. This would be a natural starting point for the definition of some higher dimensional analogues of the Gromov-Witten invariants for complex surfaces. This theory may serve as a microscopic description of topological F-theory [3].

Acknowledgments

We are grateful to Sergei Gukov, Ulf Lindström, Andrei Losev and George Thompson for discussions. In particular we thank George Thompson for reading and commenting the manuscript. The research of G.B. is supported by the European Commission RTN Program MRTN-CT-2004-005104 and by MIUR. G.B. thanks the Yukawa Institute, Kyoto where part of this work was carried out supported by the programme "Fundamental Interactions and the Early Universe". M.Z. thanks SISSA (Trieste) and Simons workshop (Stony Brook) where part of this work was carried out. The research of M.Z. was supported by VR-grant 621-2004-3177.

References

[1] S. Akbulut and S. Salur, *Calibrated manifolds and gauge theory*, math.GT/0402368.

[2] S. Akbulut and S. Salur, *Associative submanifolds of a G2 manifold*, math.GT/0412032.

[3] L. Anguelova, P. de Medeiros and A. Sinkovics, *On topological F-theory*, *JHEP* **05** (2005) 021 [hep-th/0412120].

[4] L. Anguelova, P. de Medeiros and A. Sinkovics, *Topological membrane theory from Mathai-Quillen formalism*, *Adv. Theor. Math. Phys.* **10** (2006) 713 [hep-th/0507089].

[5] L. Bao, V. Bengtsson, M. Cederwall and B.E.W. Nilsson, *Membranes for topological M-theory*, *JHEP* **01** (2006) 150 [hep-th/0507077].

[6] L. Baulieu and I.M. Singer, *The topological sigma model*, *Commun. Math. Phys.* **125** (1989) 227.

[7] L. Baulieu and A. Tanzini, *Topological symmetry of forms, $N = 1$ supersymmetry and S-duality on special manifolds*, *J. Geom. Phys.* **56** (2006) 2379 [hep-th/0412014].

[8] C. Beasley and E. Witten, *Residues and world-sheet instantons*, *JHEP* **10** (2003) 065 [hep-th/0304115].

[9] C. Beasley and E. Witten, *Non-abelian localization for Chern-Simons theory*, *J. Diff. Geom.* **70** (2005) 183 [hep-th/0503126].

[10] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158].

[11] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Springer, Berlin (1996).

[12] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, *Nucl. Phys.* **B 405** (1993) 279 [hep-th/9302103].

[13] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Topological field theory*, *Phys. Rept.* **209** (1991) 129.

[14] M. Blau and G. Thompson, *N=2 topological gauge theory, the Euler characteristic of moduli spaces and the Casson invariant*, *Commun. Math. Phys.* **152** (1993) 41 [hep-th/9112012].

[15] G. Bonelli and M. Zabzine, *From current algebras for p-branes to topological M-theory*, *JHEP* **09** (2005) 015 [hep-th/0507051].

[16] G. Bonelli, A. Tanzini and M. Zabzine, *On topological M-theory*, *Adv. Theor. Math. Phys.* **10** (2006) 239 [hep-th/0509175].

[17] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, *Gauge theory at large-N and new G_2 holonomy metrics*, *Nucl. Phys.* **B 611** (2001) 179 [hep-th/0106034].

[18] R. Bryant and S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, *Duke Math. J.* **58** (1989) 829.

[19] J. Bryan, S. Katz and N.C. Leung, *Multiple covers and the integrality conjecture for rational curves in Calabi-Yau threefolds*, math.AG/9911056.

[20] J. de Boer, A. Naqvi and A. Shomer, *The topological G_2 string*, hep-th/0506211.

[21] J. de Boer, P. de Medeiros, S. El-Showk and A. Sinkovics, *Open G_2 strings*, hep-th/0611080.

[22] R. Dijkgraaf, S. Gukov, A. Neitzke and C. Vafa, *Topological M-theory as unification of form theories of gravity*, *Adv. Theor. Math. Phys.* **9** (2005) 593 [hep-th/0411073].

[23] R. Gopakumar and C. Vafa, *M-theory and topological strings. II*, hep-th/9812127.

[24] A.A. Gerasimov and S.L. Shatashvili, *Towards integrability of topological strings. I: three-forms on Calabi-Yau manifolds*, *JHEP* **11** (2004) 074 [hep-th/0409238].

[25] P.A. Grassi and P. Vanhove, *Topological M-theory from pure spinor formalism*, *Adv. Theor. Math. Phys.* **9** (2005) 285 [hep-th/0411167].

[26] J.A. Harvey and G.W. Moore, *Superpotentials and membrane instantons*, hep-th/9907026.

[27] R. Harvey and H.B. Lawson Jr., *Calibrated geometries*, *Acta Math.* **148** (1982) 47157.

[28] N. Ikeda and T. Tokunaga, *Topological membranes with 3-form H flux on generalized geometries*, hep-th/0609098.

[29] D. Joyce, *On counting special Lagrangian homology 3-spheres*, *Contemp. Math.* **314** (2002) 125 [hep-th/9907013].

[30] A. Losev, *Perspectives of string theory*, talk at the String theory at Greater Paris seminar (2001).

[31] R.C. McLean, *Defomations of calibrated submanifolds*, *Commun. Anal. Geom.* **6** (1998) 705.

JHEP03(2007)023

[32] N. Nekrasov, *A la recherche de la m-theorie perdue. Z theory: casing m/f theory*, hep-th/0412021.

[33] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, *Nucl. Phys.* **B 577** (2000) 419 [hep-th/9912123].

[34] E. Witten, *Topological sigma models*, *Commun. Math. Phys.* **118** (1988) 411.

[35] E. Witten, *Mirror manifolds and topological field theory*, hep-th/9112056.

[36] S. Wu, *Mathai-Quillen formalism*, hep-th/0505003.

JHEP03(2007)023